Harmonia Trigonometrica;

OR, A

SHORT TREATISE

OF

TRIGONOMETRY:

WHEREIN

The *Harmony* between *Plane* and *Spherical* Trigonometry is clearly exhibited, and thereby all the Difficulties and Perplexities of the latter are entirely removed; ſo that both are render'd equally eaſy, their ſimilar Caſes being ſolved by Theorems *materially*, and almoſt *verbally* the ſame.

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THE

PREFACE.

THE chief Deſign of this Tract (as the Title Page expreſſes) is to ſhew the *Harmony* between *plane* and *ſpherical* Trigonometry. And the better to anſwer that Purpoſe, the correſpondent Theorems of both are diſpoſed in oppoſite Pages, facing each other; whereby the Agreement between them is rendered conſpicuous at one View. This Deſign indeed takes up but a ſmall Part of the Treatiſe, but the other Particulars here added are not, 'tis hoped, without their Uſe.

THE ELEMENTS OF TRIGONOMETRY.

CHAP. I.

DEFINITIONS.

I. TRIGONOMETRY is the Art of *measuring* or *resolving Triangles.*

II. A Triangle is said to be *resolved*, when from three given Parts (-either Sides, Angles*, or both-) a fourth (-either Side or Angle-) is found out.

* Except the three Angles of a plane Triangle; for these determine only the *Proportion* of the Sides, and not their *Quantity*.

III. *Trigonometry* is either *plane* or *spherical*; the first resolves *plane* Triangles, the second *spherical* ones.

SCHOLIUM.

Inasmuch as for the resolving of Triangles, it is required, that the Proportion which all the Parts of a Triangle bear to each other be known, which really is not; for the Peripheries of Circles, (and consequently the Measures of *Angles* in a *plane* Triangle, and the Measures of *Sides* and *Angles* in a *Spherical* Triangle,) could never yet be reduced to right Lines; Mathematicians therefore, in order to remedy this Defect, have applied certain right Lines to the Circle, which bear an invariable Ratio to the Radius thereof. And these Lines they called - *Chords* - *Sines* - *Tangents* and *Secants*.

IV. A *Chord* is a right Line drawn from one End of an Arc to the other; thus A G is the *Chord* of the Arc A E G, and of the Arc A H G.

COROLLARY.

The *Chord* therefore of an *Arc*, and of its *Complement* to a Circle, is the *same*.

A *Sine* is either *right* or *versed*.

V. A *Right Sine*, most commonly called, simply, a *Sine*, is a Perpendicular let fall from one Extremity of an Arc, upon the Radius drawn to the other Extremity of that same Arc; or it is one Half of the Cord of the double Arc. Thus the *Right Sine* of the *Arc* AE is the Line AC, $= \frac{1}{2}$ AG, the Chord of AEG double of AE.

COROLLARY.

As the Line AG is the Chord of the Arc AEG, and likewise of AHG its *Complement* to a Circle, consequently AC ($= \frac{1}{2}$ AG) the *Sine* of the *Arc* AE, is likewise the *Sine* of its *Supplement* AH. The *Sine* therefore, of an *Angle*, and of its *Supplement* to 180 Degrees, or two right Angles, is the *same*.

SCHOLIUM.

It appears from *Def.* 5. that every Sine (i. e. right Sine) is the half of a Chord, and that the Arc of the Sine is the half of the Arc of the said Chord. Wherefore, because there is the same Proportion between two Halves, as there is between their Wholes, and the said Proportion between the Halves (as being expressible in less Numbers) is much easier to be calculated, than the Proportion between their Wholes; hence (says Dr WELLS) Sines are used rather than Chords in Trigonometrical Calculations, and the Proportion of Sines (not of Chords) to the Radius is set forth in Trigonometrical Tables.

VI. A *versed Sine* is the Segment of the Diameter intercepted between the right Sine and the Arc. Thus CE is the *versed Sine* of the *Arc* AE, and CH the *versed Sine* of its *Supplement* AH.

VII. The *Tangent* of an Arc is a right Line drawn perpendicular to the Diameter, touching the Arc at one End, and produced till it meets

VIII. The *Secant*, which is a right Line drawn from the Center thro' the other End of the said Arc. Thus ED is the *Tangent*, and BD the *Secant* of the *Arc* AE.

SCHOLIUM.

It is to be obſerved, that as the Sine of an Arc, and of its *Supplement*, is the ſame; ſo the *Tangent* or *Secant* of any *Arc*, and of that Arc's *Supplement*, is the *ſame* likewiſe.

IX. The *Coſine*, *Cotangent*, and *Coſecant* of an *Arc*, is the *Sine*, *Tangent*, and *Secant* of that Arc which is the *Complement* of the former Arc to a Quadrant. Thus, FA, the *Sine* of AK; KL, its *Tangent*; and BL, its *Secant*; are reſpectively the *Coſine*, *Cotangent*, and *Coſecant* of the *Arc* AE.

CHAP. II.

The PROPERTIES *or* AFFECTIONS *of* Spherical *Triangles.*

DEFINITIONS.

I. A *Spherical Triangle* is that which is comprehended under three Arcs of *great Circles* interſecting each other on the Surface of the *Sphere*.

II. The *Meaſure* of a *Spherical Angle* is an Arc of a *great Circle* intercepted between the Sides comprehending the Angle, the *Pole* of that Circle being the *angular Point*.

PROPERTIES.

I. In every Spherical Triangle each Side is *leſs* than a Semicircle.

II. Any *two Sides* taken together are *greater* than the *third*.

III. The *three Sides* taken together are *leſs* than a whole Circle.

IV. The *three Angles* taken together are *greater* than *two right Angles*, and *leſs* than *ſix*.

V. The *greater Angle* is ſubtended by the *greater Side*, and the *leſſer Angle* by the *leſſer Side*.

VI. In a right-angled Spherical Triangle, the *Angles* are of the *ſame* Affection with their *oppoſite Sides*; that is, if the Sides are *equal*, *greater* or *leſs* than Quadrants, the Angles oppoſite thereto are *equal*, *greater* or *leſs* than Right Angles. And v. v.

VII. If the two *Sides* of a right-angled Spherical Triangle be of the *ſame* Affection, (and conſequently the *Angles*) that is, if they are *both leſs*, or *both greater* than a Quadrant, the *Hypotenuſe* will be *leſs* than a Quadrant.

VIII. But if the two *Sides* (and consequently the *Angles*) be of *different* Affection, i. e. if one be *less*, and the other *greater* than a Quadrant, the *Hypotenuse* will be *greater* than a Quadrant.

IX. If the *Hypotenuse* be *greater* than a Quadrant, then the *Sides* comprehending the right Angle, and also the *Angles opposite* to them, are of *different* Affection; but if *less* than a Quadrant, they are of the *same* Affection.

X. Any Spherical Triangle may be transformed into another, whose three Sides are equal to the three *Angles* of the former, or their *Supplements*, if any of them be *obtuse*. And contrariwise, whose *Angles* are equal to the *Sides* of the former, or their *Supplements* to Semicircles, if they are *greater* than Quadrants.

C O R O L L A R Y.

Hence it follows, that, when the three *Angles* of a Spherical Triangle are given, the three *Sides* of the reciprocal Triangle will be known; for the two *lesser Sides* will be respectively equal to the two *lesser*, or *acute Angles*, and the *greatest Side* to the *Supplement* of the *greatest*, or *obtuse Angle*.

C H A P. III.

IN every *right-angled plane Triangle*, any of the three Sides may be made *Radius*, and then the two other Sides will be as *Sines*, *Tangents*, or *Secants*, as appears from the following

T H E O R E M.

If the *Hypotenuse* be made Radius, the other two *Sides* are the *Sines* of their *opposite Angles*; but if *one* of the *Sides* (i. e. the *Base* or *Perpendicular*) be made Radius, then the *other Side* is the *Tangent* of its *opposite Angle*, and the *Hypotenuse* is the *Secant* of that *same Angle*.

This will appear very clearly, if the Figures in the *Synopsis* be compared with *Def*. V. VII. VIII.

S C H O L I U M.

It may be of some use, to apprize the Reader in this Place, that there is in the *Trigonometrical Canon*, or *Tables*, a Triangle exactly *similar* to the Triangle proposed to be solved; and upon the *Similarity* of these Triangles all Trigonometrical Operations are grounded. Wherefore the Triangle proposed to be resolved is always considered in a *double Respect*:

—1. As

— 1. As the *Given* Triangle — and 2. As the *Canonical* Triangle — and then *each Side* (as it is obſervable in the forementioned Figures) has *two* Denominations; the one *fixed* and *unchangeable*, viz. *Hypotenuſe*, *Baſe* and *Perpendicular*; the other continually *changing*, according to the Side made *Radius*. But the Denomination of any Side (under this ſecond Conſideration) may be readily known by the Word annexed to it.

CHAP.

CHAP. IV.

The Trigonometrical Solutions of right-angled plane *Triangles.*

THE *Cases* of *right-angled* Triangles are *six*; all which may be solved by the following *Theorems.*

THEOREM I.

As Radius,
Is to the Hypotenuse;
So is the Sine of either of the oblique Angles,
To its opposite Side.

viz. R : AC : : S.C : AB. Fig. 1.

THEOREM II.

As Radius,
Is to the Base;
So is the Tangent of the adjacent Angle,
To the Perpendicular.

viz. R : BC : : T.C : AB. Fig. 2.

SCHOLIUM.

By introducing *Secants* into the Analogy, we have this additional *Theorem* for the Solution of all right-angled Triangles, *viz.*

As the Base,
Is to Radius;
So is the Hypotenuse,
To the Secant of the adjacent Angle. *See Fig.* 2, 3.

CHAP. IV.

The Trigonometrical Solutions of right-angled Spherical *Triangles.*

THE *Caſes* of *right-angled* Triangles are *ten*; all which may be ſolved by the two following *Theorems.*

THEOREM. I.

As Radius,
Is to the Sine of the Hypotenuſe;
So is the Sine of either of the oblique Angles,
To the Sine of its oppoſite Side.

viz. R : S.AC :: S.C : S.AB. Fig. 5.

COROLLARY.

Hence it follows, that, if

1. An *Angle* and its *oppoſite Side*	be given	The *Hypotenuſe*	may be found.
2. An *Angle* and the *Hypot.*		The *oppoſite Side*	
3. The *Hypot.* and one *Side*		The *oppoſite Angle*	

THEOREM II.

As Radius,
Is to the Sine of the Baſe;
So is the Tangent of the adjacent Angle,
To the Tangent of the Perpendicular.

viz. R : S.BC :: T.C : T.AB. Fig. 5.

COROLLARY.

Hence it follows, that, if

1. The *two Sides*	be given	The *two acute Angles*	may be found.
2. An *Angle* and the *adjacent Side*		The *other Side*	

SCHOLIUM.

Tho' there is an exact *Harmony* between *plane* and *ſpherical* Trigonometry, with regard to the *Theorems* whereby this *Species* of Triangles is reſolved; yet it muſt ſtill be confeſſed, that, with regard to the *Solution*, there is this *ſmall Difference* between them; *viz.* that in reſolving *plane Triangles* we may uſe either *Sines* or *Tangents* (i. e. Theorem 1. or 2.) indifferently; whereas in *ſpherical Trigonometry*, the Caſe that is ſolvable by *Sines* will not admit of a Solution by *Tangents*, nor that which is ſolvable by *Tangents* of a Solution by *Sines*.

CHAP.

CHAP. V.

Of the Solution *of* right-angled *Triangles, by the five Circular Parts.*

DEFINITIONS.

I. EVERY Triangle consists of *six* Parts, viz. *three Sides* and *three Angles*; but omitting the *right Angle*, as being always known, the *five* remaining Parts, *viz.* the *Hypotenuse*, the *two oblique Angles*, and the *Complements* of the *two Sides*, are called *Circular Parts.*

II. In the Resolution of every Triangle, *three* of these Parts come always under Consideration, *viz.* the *two* Parts *given*, and the *third required*; that which is situated in the *Middle*, between the other two, is called the *Middle Part*; and the other two, between which it is situated, are called *Extreme Parts.*

III. When the *Extreme Parts* lie *contiguous*, or are *conjoined* to the *Middle Part*, they are called *Extremes Conjunct*:

IV. But when they are *disjoined* from it, that is, when *another Part* interposeth on *both Sides* between the *Extreme Parts* and the *Middle Part*, then they are called *Extremes Disjunct.*

COROLLARY.

Therefore, if

be the middle Part	then will be Extremes conjunct, and	Extremes disjunct.
1. ɔAB	1. ɔBC, A	1. AC, C
2. A	2. AC, ɔAB	2. ɔBC, C
3. AC	3. A, C	3. ɔAB, ɔBC
4. C	4. AC, ɔBC	4. ɔAB, A
5. ɔBC	5. C, ɔAB	5. AC, A

NB. ɔAB, &c. signifies the *Complement* of AB, &c.

SCHOLIUM.

The Right Angle being always known, is therefore ejected out of the Circular Parts, and the Sides by which it is comprehended are considered as immediately adjoining: and hence it comes to pass, that tho' the right Angle stands between BC (Case 1.) and AB (Case 5.) and the middle Part, yet they are supposed to be really conjunct. These Things premised, all the Cases of right-angled Triangles may be easily and expeditiously solved by the following *Theorems.*

THEO-

THEOREM I.

In all right-angled Spherical Triangles, the Rectangle under the *Radius*, and the *Cosine* of the *middle Part*, is equal to the Rectangle under the *Sines* of the *Extremes disjunct.*

COROLLARY.

Hence it is evident, that, if from the *Sum* of the Logarithms of the *Radius* and the *Cosine* of the *middle Part*, you *subtract* the Logarithm of the *Sine* of *either* of the *Extremes disjunct*, the *Remainder* will be the Logarithm of the *Sine* of the *other*.

THEOREM II.

The Rectangle under the *Radius*, and the *Cosine* of the *middle Part*, is equal to the Rectangle under the *Cotangents* of the *Extremes conjunct.*

COROLLARY.

Hence it follows, that, if from the *Sum* of the Logarithms of the *Radius* and the *Cosine* of the *middle Part*, you *subtract* the Logarithm of the *Cotangent* of *either* of the *Extremes conjunct*, the *Remainder* will be the Logarithm of the *Cotangent* of the *other*.

SCHOLIUM I.

When a Complement happens to *concur* with a Complement in the circular Parts, then the *Sine*, or the *Tangent*, instead of the *Cosine* or *Cotangent*, is always to be used in the Proportion; for the *Cosine* or *Cotangent* of the *Complement* of an Arc, is the *Sine* or *Tangent* of *that Arc*.

SCHOLIUM II.

If we consider the *Sides* of *plane* Triangles as the *Sines* or *Tangents* of the *Sides* of *spherical* ones, and apply to *those* what has been here advanced of *these*, the Harmony between both will be still conspicuous. *See* Wolfius's Trigonometry.

SCHOLIUM III.

The Lord NAPIER, who was the first Inventor of this Method, makes use of the *Sides themselves*, and for the *Hypotenuse* and *oblique Angles* inserts their *Complements* in the circular Parts; whence he deduces these *Theorems*.

CHAP. VI.

The Trigonometrical Solutions *of* oblique-angled plane *Triangles.*

THE *Cases* of *oblique-angled* Triangles are *four*, all which may be solved by the following *Theorems.*

THEOREM I.

As one Side,
Is to the other;
So is the Sine of the Angle opposite to the first Side,
To the Sine of the Angle opposite to the other.

viz. AB : AC :: S.C : S.B. Fig. 4.

COROLLARY.

Hence it appears that, if

1. Two *Sides* and the *Angle oppos.* to one of them	be given,	1. The *Angle opposite* to the other	may be found.
2. Two *Angles* and the *Side oppos.* to one of them		2. The *Side opposite* to the other	

LEM-

THEOREM I.

The Rectangle under the *Radius*, and the *Sine* of the *middle Part*, is equal to the Rectangle under the *Cosines* of the *Extremes disjunct*.

THEOREM II.

The Rectangle under the *Radius* and the *Sine* of the *middle Part*, is equal to the Rectangle under the *Tangents* of the *Extremes conjunct*.

CHAP. VI.

The Trigonometrical Solutions *of* oblique-angled Spherical *Triangles.*

THE *Cases* of *oblique-angled* Triangles are *six*; all which may be solved by the following *Theorems.*

THEOREM I.

As the Sine of one Side,
Is to the Sine of the other;
So is the Sine of the Angle opposite to the first Side,
To the Sine of the Angle opposite to the other.

viz. S.AB : S.AC :: S.C : S.B. Fig. 6.

COROLLARY.

Hence it appears that if

<table>
<tr><td>1. Two Sides and the Angle oppos. to one of them
2. Two Angles and the Side oppos. to one of them</td><td>be given,</td><td>1. The Angle opposite to the other
2. The Side opposite to the other</td><td>may be found.</td></tr>
</table>

LEMMA.

If any one Angle of a plane Triangle be subtracted from 180 Degrees or 2 Right Angles, the Remainder will be the *Sum* of the other two Angles, (32. El. 1. Cor.) which divided by 2 gives their *Half Sum* in the Quotient.

THEOREM II.

As Half the Sum of two Sides,
Is to Half their Difference;
So is the Tangent of Half the Sum of their opposite Angles,
To the Tangent of Half their Difference.

viz. $\frac{BC+AB}{2} : \frac{BC-AB}{2} :: T.\frac{A+C}{2} : T.\frac{A-C}{2}$

Fig. 4.

COROLLARY.

From hence it follows, that, if *two Sides* and the *Angle included between them* be given, each of the other two *Angles* may be found; for their *Half Sum* will be found by *Lemma*, and their *Half Difference* by *Theorem* 2. from whence (by *Problem* p. 20.) the Angles *themselves* may be found.

SCHOLIUM. I.

Instead of *Theorem* II. the following may be used, viz.

1. As the lesser Side,
   Is to the greater;
   So is Radius,
   To the Tangent of an Arc.
2. As Radius,
   Is to the Tangent of that Arc, less 45°;
   So is the Tangent of Half the Sum of their opposite Angles,
   To the Tangent of Half their Difference.

SCHO-

LEMMA.

As the Cosine of Half the Sum of two Sides,
Is to the Cosine of Half their Difference;
So is the Cotangent of Half the included Angle,
To the Tangent of Half the Sum of their opposite Angles.

$$\text{viz. Cos. } \frac{BC+AB}{2} : \text{Cos. } \frac{BC-AB}{2} :: \text{Cot. } \frac{B}{2} : \text{T. } \frac{A+C}{2}$$

Fig. 6.

THEOREM II.

As the Tangent of Half the Sum of two Sides,
Is to the Tangent of Half their Difference;
So is the Tangent of Half the Sum of their opposite Angles,
To the Tangent of Half their Difference.*

$$\text{viz. T. } \frac{BC+AB}{2} : \text{T. } \frac{BC-AB}{2} :: \text{T. } \frac{A+C}{2} : \text{T. } \frac{A-C}{2}$$

Fig. 6.

* i.e. when the Sum of the two Sides is *less* than a *Semicircle*; when *greater*, take their *Supplements*, and the Operation will produce the *Supplements* of the Angles sought to two right Angles.

COROLLARY.

Hence it follows, that, if *two Sides* and the *included Angle* be given, each of the other two *Angles* may be known, for their *Half Sum* will be found by the *Lemma*, and their *Half Difference* by the *Theorem*, from whence (by *Problem* p. 20.) the Angles *themselves* will be known.

SCHOLIUM.

Instead of *Theorem* II. the following may be used, viz.

1. As the Sine of the lesser Side,
   Is to the Sine of the greater;
   So is Radius,
   To the Tangent of an Arc.
2. As Radius,
   Is to the Tangent of that Arc, less 45°;
   So is the Tangent of Half the Sum of their opposite Angles,
   To the Tangent of Half their Difference.

From

SCHOLIUM II.

This Theorem is purpofely inferted on account of its great Ufe i Aftronomy.

From the greateft Angle of an oblique Triangle let fall a Perpendi cular on the Bafe, dividing it into two Segments, then

THEOREM III.

As Half the Bafe
Is to Half the Sum of the other two Sides;
So is Half the Difference of thofe Sides,
To Half the Difference of the Segments of the Bafe.

viz. $\frac{BC}{2} : \frac{AB+AC}{2} :: \frac{AB-AC}{2} : \frac{BD-DC}{2}$ Fig.

COROLLARY.

From hence it follows, that, if the *three Sides* be given, the *thr Angles* may be found. For the *Segments* of the Bafe will be given by th *Theorem*: and now the oblique Triangle being reduced into two righ angled ones, wherein the *Bafe* and *Hypotenufe* are known, the *Angl* will thence be given by the Refolution of *Cafe* V. of *right-angl* Triangles.

SCHOLIUM.

The Fractions which commonly happen in the Segments may cau an Error of fome Minutes in the Angles; wherefore the following *The rem* is to be preferred (both for *Accuracy* and *Expedition)* in the Solu tion of this Cafe.

THEOREM.

As the Rectangle under the Sides comprehending the Ang fought,
Is to the Square of the Radius;

From the greateſt Angle of an oblique Triangle, let fall a Perpendicular on the Baſe, which will divide it into two Segments; then,

THEOREM III.

As the Tangent of Half the Baſe,
Is to the Tangent of Half the Sum of the other two Sides;
So is the Tangent of Half the Difference of thoſe Sides,
To the Tangent of Half the Difference of the Segments of the Baſe.

viz. $T\frac{BC}{2} : T\frac{AB+AC}{2} :: T\frac{AB-AC}{2} : T\frac{BD-DC.}{2}$ Fig. 6.

COROLLARY.

From hence it appears, that, if the *three Sides* be given, either of the *three Angles* may be found. For the *Segments* of the Baſe will be given by this *Theorem*; wherefore the oblique Triangle being reduced into two right-angled ones, wherein the *Baſe* and *Hypotenuſe* are known, the *Angles* will be given by the Reſolution of *Caſe* III. of *right-angled* Triangles.

SCHOLIUM.

As the Reſolution of this Caſe by the foregoing Method is very tedious, we ſhall therefore add the following *Theorem*; by which it may be ſolved at one Operation.

THEOREM.

As the Rectangle under the Sines of the Sides comprehending the Angle ſought,
Is to the Square of the Radius;

So

So is the Rectangle under the Differences between those Sides and th
Half Sum of the three Sides,

To the Square of the Sine of Half the Angle sought.

viz. $AB \times AC : Rq :: X \times Z : Sq. \frac{A}{2}$ Fig. 4.

N.B. $\frac{BC+AB+AC}{2}\left\{\begin{array}{l}-AC=X.\\-AB=Z.\end{array}\right.$

PROBLEM.

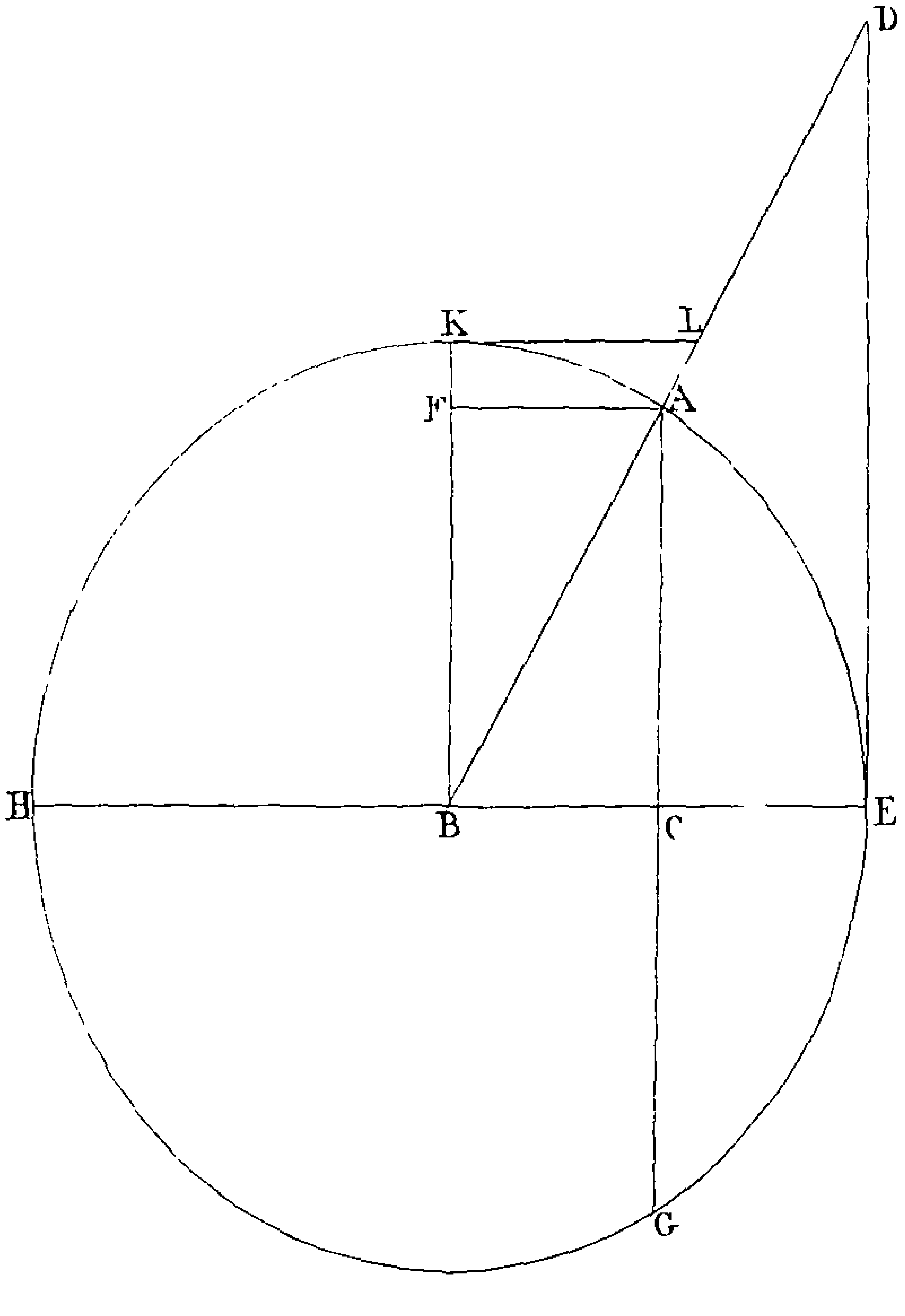
The Half Sum and Half Difference of any two Quantities bein
given, to find the Quantities themselves.

SOLUTION.

1. To Half the Sum *add* Half the Difference, the Aggregate (Sum
of both) will be the *greater* Quantity.

2. From Half the Sum *subtract* Half the Difference, the Remainder
will be the *lesser* Quantity.

CHAP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So is the Rectangle under the Sines of the Differences between those Sides and the Half Sum of the three Sides,

To the Square of the Sine of Half the Angle sought.

viz. $S.AB \times S.AC : Rq :: S.X \times S.Z : Sq. \frac{A}{2}$ Fig. 5.

N.B. $\frac{BC+AB+AC}{2}\left\{\begin{array}{l}-AC=X.\\-AB=Z.\end{array}\right.$

LEMMA.

As the Cosine of Half the Sum of two Angles,
Is to the Cosine of Half their Difference;
So is the Tangent of Half the interjacent Side,
To the Tangent of Half the Sum of the other two Sides.*

viz. $Cos. \frac{C+B}{2} : Cos. \frac{C-B}{2} :: T. \frac{BC}{2} : T. \frac{AB+AC}{2}$ Fig. 6.

* i. e. When the SUM of the Angles is *less* than two *right Angles* — when *greater*, take their *Supplements*, and the Operation will produce each Side's *Supplement* to a Semicircle.

THEOREM IV.

As the Tangent of Half the Sum of two Angles,
Is to the Tangent of Half their Difference;
So is the Tangent of Half the Sum of their opposite Sides,
To the Tangent of Half their Difference.

viz. $T. \frac{C+B}{2} : T. \frac{C-B}{2} :: T. \frac{AB+AC}{2} : T. \frac{AB-AC}{2}$ Fig. 6.

COROLLARY.

From hence it follows, that if *two Angles* and the *interjacent Side* be given, the other two *Sides* may be found. for their *Half Sum* will be known by the *Lemma*, and their *Half Difference* by the *Theorem*; whence (by *Problem* p. 20) the Sides *themselves* will be known.

C H A P. VII.

Of the Solutions *of* Trigonometrical Problems.

IN all *Trigonometrical Operations* there are always *three* Things given, to find a *fourth*; supposing the *Radius* in *right-angled* Triangles to be one. These three given Terms are to be so stated, that the same Proportion there is between the first and second, may likewise exist between the third and fourth, which is always the Term *sought*; and therefore anciently (as in all other Cases in the Rule of Proportion) the Method was, to multiply the second and third Terms into each other, and divide the Product by the first; the Quotient was the Answer to the Question. But now since the admirable Invention of *Logarithms* (the Nature of which is such that *Addition* performs the Office of *Multiplication*, and *Subtraction* of *Division*) there needs only to *add* the *Logarithms* of the second and third Terms together, and from their Sum to *subtract* the *Logarithm* of the first Term, and the Remainder will be the *Logarithm* of the fourth Term required.

But for more Ease and Expedition,

When the Radius is not in the Proportion, then, instead of the Logarithm of the first Term, *itself*, set down its *Arithmetical Complement* (i. e. what each Figure but the last wants of 9, and that of 10.) and *add* all the *three* Terms together, the Sum, (subtracting Radius,) is the *Logarithm* of the fourth Term sought.

If the Logarithm of the Term sought should not precisely correspond with any Logarithm in the Tables, it is sufficient for ordinary Purposes to take the nearest to it. But where great Exactness is required, you must proportion the Difference as the Nature of the Case demands.

In stating the Terms of the Analogies, you are carefully to observe the following

P R E C E P T S.

I. When a *Side* is required, begin with an *Angle*; and, on the contrary, when an *Angle* is required, begin with a *Side*; then,

II. Compare a *Side* to its *opposite Angle*, and an *Angle* to its *opposite Side*.

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By an Angle I mean the *Sine*, *Tangent*, or *Secant* of it. By a Side, (in Sph. Trigon.) the *Sine* or *Tangent* of that Side.

PROBLEM I.

In a *right-angled plane* Triangle, any *two Parts* (as the *Side* BC, and the *Angle* A) being given, to find the *other Parts*. V. *Synopsis*, *p.* 28.

SOLUTION.

I. Consider which Side is most proper to be made Radius, whether the *Hypotenuse*, *Base*, or *Perpendicular*.

II. If the *Hypotenuse* be made Radius, (Fig. 1.) the Analogy will be

1. S.A : BC :: R : AC.
2. S.A : BC :: S.C : AB.

III. If the *Base* be made Radius, (Fig. 2.) the Proportion is

1. R : BC :: T.C : AB.
2. R : BC :: Sec.C : AC.

IV. If you make the *Perpendicular* Radius, (Fig. 3.) the Solution is performed by the following Analogy, viz.

1. T.A : BC :: R : AB.
2. T.A : BC :: Sec.A : AC.

PROBLEM II.

In a *right-angled Spherical* Triangle, any *two Parts* (besides the right Angle) being given to find the *other Parts*. V. *Synopsis*, *p.* 30.

The SOLUTION *by the* common Rules.

I. First consider whether the *Parts* concerned in the Question be *Extremes conjunct* or *disjunct*.

II. If the *Parts* be *Extremes disjunct*, and *opposite* to each other;

e. g. if the *Hypotenuse* AC and the *Angle* C be given, to find its *opposite Side* AB; then say by *Theorem* I.

As Radius,
Is to the Sine of the Hypotenuse AC;
So is the Sine of the Angle C,
To the Sine of its opposite Side AB.

III. But if the *Parts* be *Extremes disjunct*, and *not* opposite; for Example, if the *Side* BC, and the *adjacent Angle* C, be given, to find the *other Angle* A: Then the Sides of the Triangle are to be continued till they become Quadrants, and thereby form a *new* Triangle, wherein the *Parts* concerned in the Problem are *Extremes disjunct*, and *opposite*, as the Triangle FCG; in which are given GC, the *Complement* of the Side BC, and the *Angle* C, to find GF, the *Complement* of the Angle A; therefore say as before,

As Radius,
Is to the Sine of the Hypotenuse GC. i. e. the Cos. of BC;
So is the Sine of the Angle C,
To the Sine of its opposite Side FG. i. e. the Cos. of A.

IV. If the *Parts* be *Extremes conjunct*, and the *Hypotenuse out* of the Question; for instance, if the *Sides* AB and BC, be given, to find the *Angle* C, the Rule is,

As the Sine of BC,
Is to the Radius;
So is the Tangent of AB,
To the Tangent of C. *Theorem* 2*d*.

V. But if the *Parts* be *Extremes conjunct*, and the *Hypotenuse* concerned *in* the Problem; as if the *Hypotenuse* AC, and the *Angle* C, were given, to find the *adjacent Side* BC: Produce each Side of the Triangle to a Quadrant, in order to find a *new* Triangle, wherein the *Hypotenuse* is *excluded out of* the Problem, as the Triangle EAK, wherein are given EA, the *Complement* of the Hypotenuse AC; EK, the *Complement* of the Angle C; to find the *Angle* K, the *Complement* of the Side sought BC; therefore the Rule is the same as before,

As the Sine of EK, viz. Cos. of C,
Is to the Radius;
So is the Tangent of EA, viz. Cot. of AC,
To the Tangent of K, viz. Cot. of BC.

VI. When the Sides of a Triangle are to be produced, it matters not towards which Part you produce them, if *neither* of the *acute Angles* are concerned in the Question. When *one* is concerned, they must always be produced thro' the *other* Angle; but when *both* the Angles en-

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ter the Problem, the Sides are to be produced thro' *that* Angle which is *adjacent* to the Side in Queſtion.

The SOLUTION *by the* Catholic Propoſition.

I. Firſt conſider, as before, whether the *Parts* in Queſtion be *Extremes conjunct* or *disjunct*.

II. If *one* or *both* the *Sides* comprehending the right Angle are concerned in the Problem, then, inſtead of the Sides *themſelves*, ſet down their *Complements* to a Quadrant; wherefore,

III. Since, by the *Catholic Propoſition*, the *Rectangle* under the *Radius* and the *Coſine* of the *Middle Part*, is equal to the *Rectangle* under the *Sines* of the *Extremes disjunct* — and the *Cotangents* of the *Extremes conjunct:* If from the *Sum* of the Logarithms of *two Parts* you *ſubtract* the Logarithm of the *third*, the *Remainder* will be the Logarithm of the *Sine* or *Tangent* of the *Side* or *Angle* ſought. *See Corol. to Th.* I. and II. p. 13.

PROBLEM III.

In the Reſolution of *right-angled* Spherical Triangles, to determine the *Species* of the *Side* or *Angle* found.

SOLUTION.

1. When the *Hypotenuſe* and *either* of the *Angles* are given, the Species of the *other Parts* may be determined by *Properties* IX and VI. Therefore *Caſe* I. and II. are free of all Ambiguity.

2. If the *Hypotenuſe* and *either Side* be given, as in *Caſe* III. and IV. the Affection of the *other Parts* is determined, as before, by *Properties* IX. and VI.

3. When *both* the *Sides*, or *both* the *Angles* are given, the Species of the *other Parts* are known by *Properties* VI. VII. VIII. Hence *Caſes* IX. and X. are clear.

4. If a *Side* (*Leg*) and an *Angle adjacent* thereto be given, as in *Caſe* V. and VII. then the Species of the *oppoſite Side* may be determined by *Property* VI. and from thence the Species of the *Hypotenuſe* by *Properties* VII. and VIII.

5. But when a *Side* and an *Angle opposite* thereto are given, then the Species of the *unknown Parts* cannot be determined. Therefore Cases VI. and VIII. are wholly *ambiguous.*

PROBLEM IV.

In the Solution of *oblique-angled* Spherical Triangles, to determine the *Species* of the *Side* or *Angle* found.

SOLUTION.

1. When two *Angles* and a *Side opposite* to *one* of them are given, to find the *Side opposite* to the *other*, as in *Case* I. proceed thus: Add both the given *Angles* together, and observe their Sum; then to the given *Side* add the *lesser* Arc of the *Side* found, and also its *greater* Arc, or *Complement*; and *that* Arc of the *Side found* which, together with the *Side given*, is of the *same* Affection with the Sum of the *Angles* (i. e. whose Sum is either *greater* or *less* than a Semicircle, as the Sum of the Angles is *greater* or *less* than two right Angles) is the *true* Arc required.

But if the Sum of the Side given and *either* Arc of the Side found be of the *same* Species with the Sum of the Angles, then the Case is *ambiguous.*

2. In like manner may the Ambiguity in *Case* IV. (where two *Sides* and an *Angle opposite* to *one* of them are given, to find the *Angle opposite* to the *other*) be cleared and determined — for *that* Value (whether the *greater* or *lesser*) of the *Angle found*, which, together with the *Angle given*, is of the *same* Affection with the Sum of the *Sides* (i. e. whose Sum is *greater* or *less* than two right Angles, as the Sum of the *Sides* is *greater* or *less* than a Semicircle) is the *true* Value required.

But if the Sum of the Angle given and *either* Value of the Angle found, be of the *same* Species with the Sum of the Sides, then the Case is *ambiguous.*

3. The other Cases, when solved by the Method here proposed, are clear of all Ambiguities.

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But it may not be improper to remark, that a Caſe which is ambiguous in *Trigonometry*, is very often not ſo, when it occurs in *Geography* or *Aſtronomy*, &c. for the peculiar Nature of the Subject to which it is applied, may ſometimes determine it, when the Principles of Trigonometry cannot.

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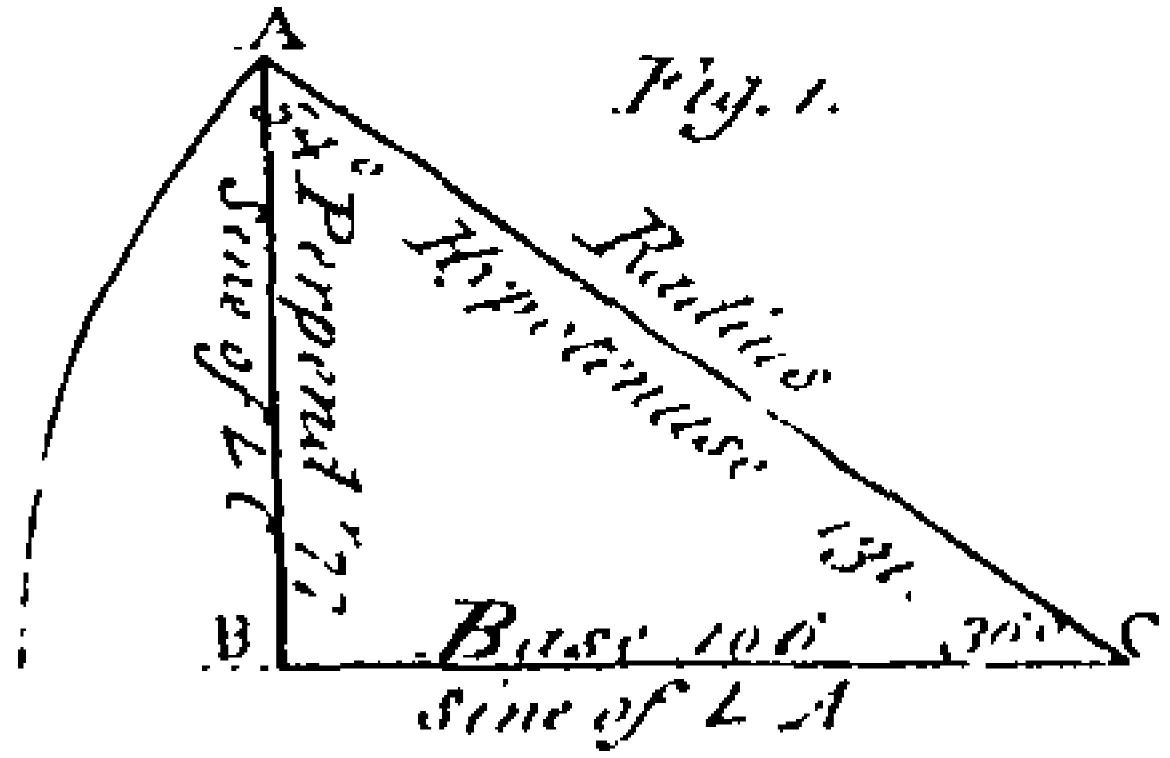
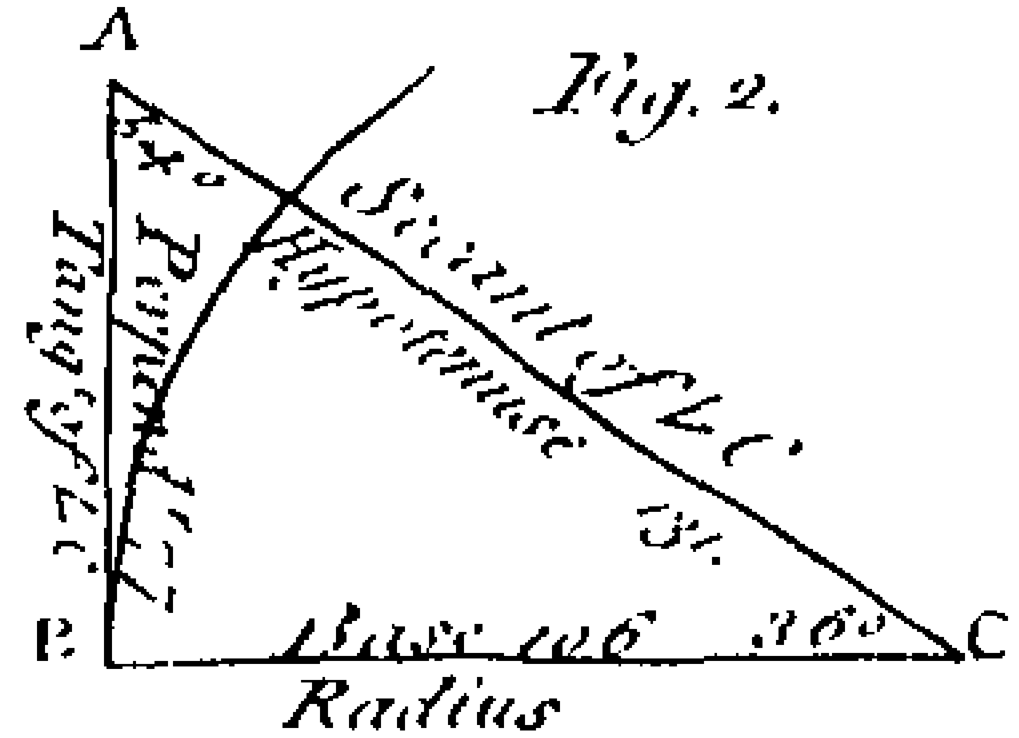
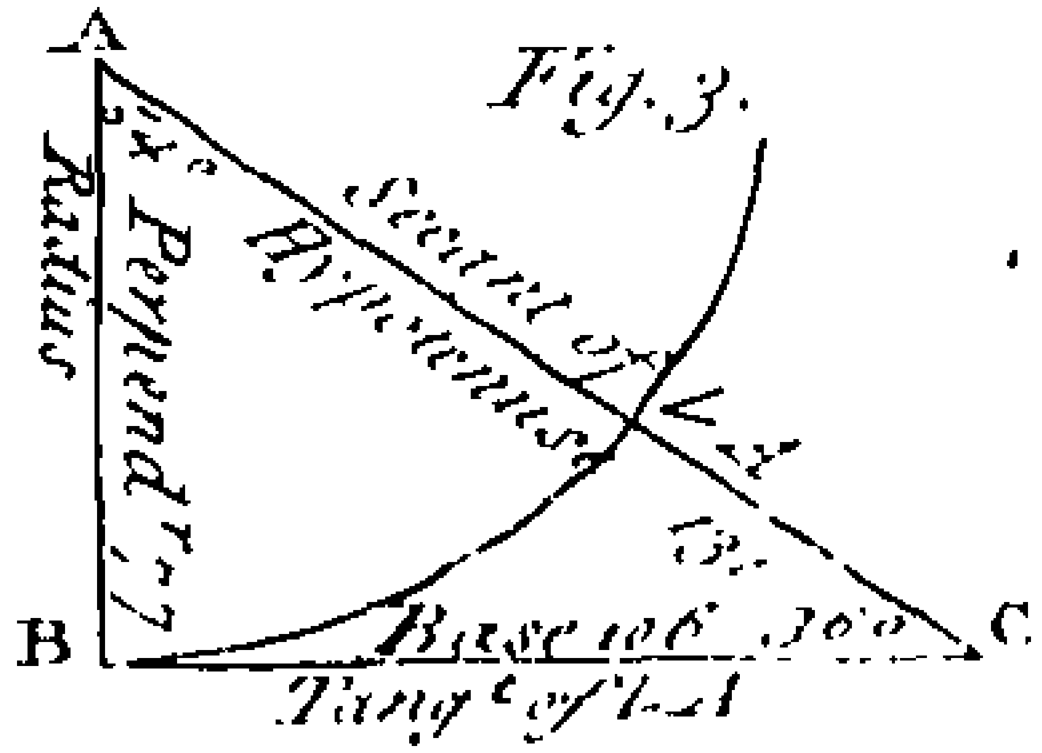
A SYNOPSIS *of the Doctrine of* right-angled *plane Triangles.*







Cases	Given	Sought	The Hypotenuſe Radius.	The Baſe Radius.	The Perpendicular Radius.
I	B C	A B	S.A : BC : : S.C : AB	R : BC . : T.C : AB	T.A : BC : : R : AB
	A.C	A C	S.A : BC : : R : AC	R : BC : : Sec.C : AC	T.A : BC : : Sec.A : AC
II	A B	B C	S.C : AB : : S.A : BC	T.C : AB : : R : BC	R : AB : : T.A : BC
	A.C	A C	S.C : AB : : R : AC	T.C : AB : : Sec. C : AC	R : AB : : Sec.A : AC
III	A C	B C	R : AC : : S.A : BC	Sec.C : AC : : R : BC	Sec.A : AC : : T.A : BC
	A.C	A B	R : AC : : S.C : AB	Sec.C : AC : : T.C : AB	Sec.A : AC : : R : AB
IV	B C	A.C		BC : R : : AB : T.C then	AB : R : : BC : T.A then
	A B	A C		R : BC : : Sec.C : AC	R : AB : : Sec.A : AC
V	A C	A.C	AC : R : : BC : S.A then	BC : R : : AC : Sec.C then	
	B C	A B	R : AC : : S.C : AB	R : BC : T C : AB	
VI	A C	A.C	AC : R : : AB : S.C then		AB : R : : AC : Sec A then
	A B	B C	R : AC : : S.A : BC		R : AB : : T.A ; BC

SYNOPSIS *of the Doctrine of* oblique-angled *plane Triangles.*

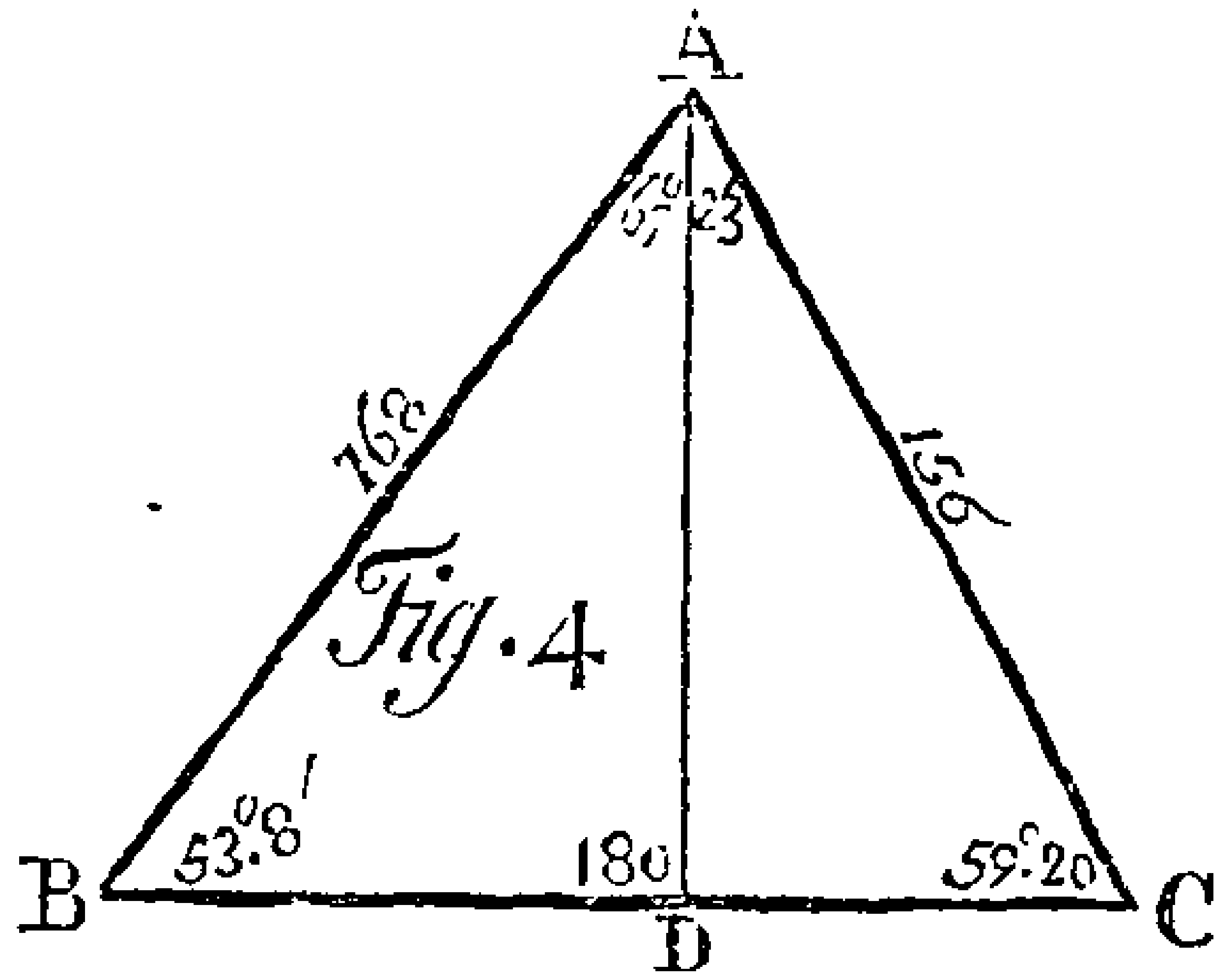


Fig. 4

Cases	Given	Sought	
I	A.B.C BC	A B, A C	S.A : BC :: S.C : AB S.A : BC :: S.B : AC
II	AB.AC B	A.C BC	AC : S.B :: AB : S.C. then 180—B—C=A. then S.B : AC :: S.A : BC.
III	AB.AC A	B.C BC	$\frac{AB+AC}{2} : \frac{AB-AC}{2} :: T.\frac{C+B}{2} : T.\frac{C-B}{2}$ then S.B : AC :: S.A : BC.
IV	AB BC AC	A B C	$\frac{BC}{2} : \frac{AB+AC}{2} :: \frac{AB-AC}{2} : \frac{BD-DC}{2}$ hence BD, DC are found, and the Angles by Case V. of right-angled Triangles. Otherwise thus. $AB \times AC : Rq :: X \times Z : Sq\ \frac{A}{2}$ and so for any other. *N.B.* $\frac{BC+AB+AC}{2} \begin{cases} -AC=X \\ -AB=Z \end{cases}$

A SYNOPSIS *of the Doctrine of* right-angled *Spherical Tria*

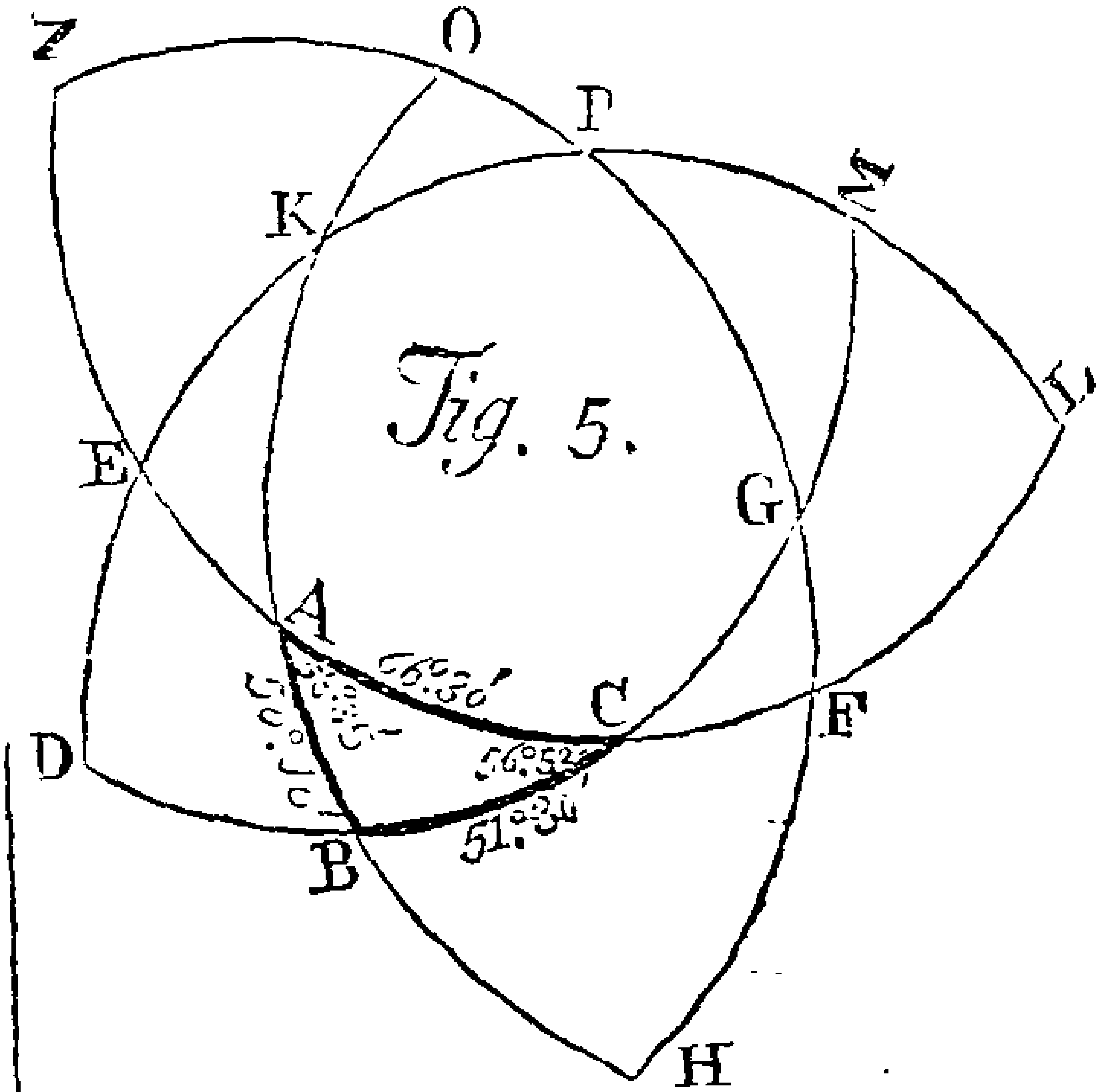
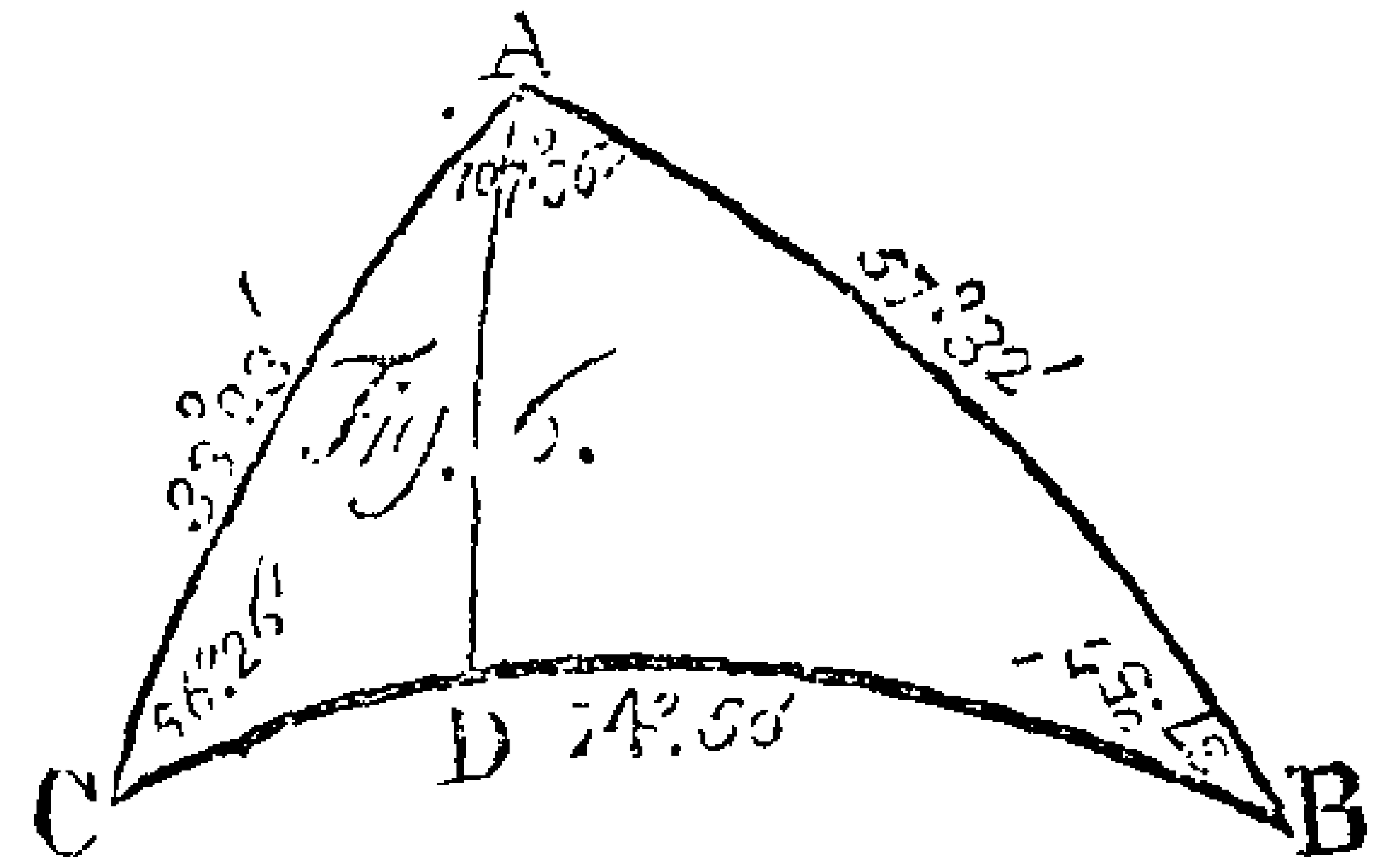


Fig. 5.

Cases	Parts given	Parts sought	Quadrantal Triangles		
I	AC A	BC AB C	AHF GHB CLM	S.AF : S.AC :: S.FH : S.BC S.GF : S.GH :: T.CF : T.BH S.CF : S.CL :: T.GF : T.LM	R : S.AC :: S.A : S.BC Cof.A : R :: Cot.AC : Cot. AB Cof.AC : R :: Cot.A : T.C
II	AC C	AB BC A	EDC KDB CLM	S.EC : S.AC :: S.ED : S.AB S.KE : S.KD :: T.EA : T.DB S.CL : S.CF :: T.LM : T.FG	R : S.AC :: S.C : S.AB Cof.C : R :: Cot.AC : Cot.BC R : Cof.AC :: T.C : Cot.A
III	AC BC	AB A C	GHB AHF KDE	S.GC : S.GB :: S.CF : S.BH S.AC : S.AF :: S.BC : S.HF T.DB : T.EA :: S.KD : S.KE	Cof.BC : R :: Cof.AC : Cof.AB S.AC : R :: S.BC : S.A Cot.BC : Cot.AC :: R : Cof.C
IV	AC AB	BC C A	KDB CDE GHB	S.KA : S.KB :: S.AE : S.BD S.AC : S.EC :: S.AB : S.ED T.BH : T.CF :: S.GH : S.GF	Cof.AB : R :: Cof.AC : Cof.BC S.AC : R :: S.AB : S.C Cot.AB : Cot.AC :: R : Cof.A
V	BC C	AC AB A	KDB CDE CLM	S.KD : S.KE :: T.DB : T.EA S.CD : S.CB :: T.ED : T.AB S.CM : S.CG :: S.ML : S.GF	R : Cof.C :: Cot.BC : Cot.AC R : S.BC :: T.C : T.AB R : Cof.BC :: S.C : Cof.A
VI	BC A	AC AB C	AHF AHF CLM	S.HF : S.BC :: S.AF : S.AC T.HF : T.BC :: S.AH : S.AB S.CG : S.CM :: S.GF : S.ML	S.A : S.BC :: R : S.AC T.A : T.BC :: R : S.AB Cof.BC : R :: Cof.A : S.C
VII	AB A	AC BC C	GHB AHF ANO	S.GH : S.GF :: T.HB : T.FG S.AH : S.AB :: T.HF : T.BC S.AO : S.AK :: S.ON : S.KE	R : Cof.A :: Cot.AB : Cot.AC R : S.AB :: T.A : T.AC R : Cof.AB :: S.A : Cof.C
VIII	AB C	AC BC A	CDE CDE ANO	S.ED : S.AB :: S.EC : S.AC T.ED : T.AB :: S.DC : S.BC S.AK : S.AO :: S.KE : S.ON	S.C : S.AB :: R : S.AC T.C : T.AB :: R : S.BC Cof.AB : R :: Cof.C : S.A
IX	AB BC	AC A C	KDB AHF CDE	S.KB : S.KA :: S.BD : S.AE S.AB : S.AH :: T.BC : T.HF S.BC : S.DC :: T.AB : T.DE	R : Cof.AB :: Cof.BC : Cof.AC S.AB : R :: T.BC : T.A S.BC : R :: T.AB : T.C
X	A C	AC BC AB	CLM CLM ANO	T.LM : T.FG :: S.CL : S.CF S.LM : S.FG :: S.CM : S.CG S.ON : S.KE :: S.AO : S.AK	T.C : Cot.A :: R : Cof.AC S.C : Cof.A :: R : Cof.BC S.A : Cof.G :: R : Cof.AB

SYNOPSIS *of the Doctrine of oblique-angled Spherical Triangles.*



Cases	Given	Sought	
I	B	AC	S.C : S.AB :: S.B : S.AC. then
	C	A	$\text{Cos.}\frac{AB-AC}{2} : \text{Cos.}\frac{AB+AC}{2} :: T.\frac{C+B}{2} : \text{Cot.}\frac{A}{2}$ *
	AB	BC	$\text{Cos.}\frac{C-B}{2} : \text{Cos.}\frac{C+B}{2} :: T.\frac{AB+AC}{2} : T.\frac{BC}{2}$ ‡
II	B	AB }	$\text{Cos.}\frac{C+B}{2} : \text{Cos.}\frac{C-B}{2} :: T.\frac{BC}{2} : T.\frac{AB+AC}{2}$ again
	C	AC }	$T.\frac{C+B}{2} : T.\frac{C-B}{2} :: T.\frac{AB+AC}{2} : T.\frac{AB-AC}{2}$
	BC	A	S.AB : S.C :: S.BC : S.A.
III	AB	A }	$\text{Cos.}\frac{BC+AB}{2} : \text{Cos.}\frac{BC-AB}{2} :: \text{Cot.}\frac{B}{2} : T.\frac{A+C}{2}$ then
	BC	C }	$T.\frac{BC+AB}{2} : T.\frac{BC-AB}{2} :: T.\frac{A+C}{2} : T.\frac{A-C}{2}$
	B	AC	S.C : S.AB :: S.B : S.AC
IV	AB	B	S.AB : S.C :: S.AC : S.B. then
	AC C	A } BC }	are found as in *Case* I.
V	AB BC AC	A B C	$T.\frac{BC}{2} : T.\frac{AB+AC}{2} :: T.\frac{AB-AC}{2} : T.\frac{BD-DC}{2}$ hence BD . DC are found, and the Angles by *Case* III. of right-angled Triangles . . . Or thus $S.AB \times S.AC : Rq :: S.X \times S.Z : Sq\,\frac{A}{2}$ and so for any other. *N.B.* $\frac{BC+AB+AC}{2}\begin{cases}-AC=X\\-AB=Z\end{cases}$
VI	A B C	AB BC AC	Convert the Angles into Sides by *Cor.* to *Property* X. *page* 8. and proceed as in *Case* V.

* Lemma Th. II. ‡ Lemma Th. IV.